The Coefficients of a Fibonacci Power Series

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Consider the infinite product

$$\begin{aligned} A(x) &= \prod_{k\geq 2}(1-x^{F_k}) = (1-x)(1-x^2)(1-x^3)(1-x^5)(1-x^8)\cdots \\ &= 1-x-x^2+x^4+x^7-x^8+x^{11}-x^{12}-x^{13}+x^{14}+x^{18}+\cdots \end{aligned}$$

regarded as a formal power series. In [4], N. Robbins proved that the coefficients of $A(x)$ are all equal to $-1, 0$ or 1. We shall give a short proof of this fact, and a very simple recursive description of the coefficients of $A(x)$.

Following the notation of [4], let $a(m)$ be the coefficient of x^m in $A(x)$. It is clear that $a(m) = r_E(m) - r_O(m)$, where $r_E(m)$ is equal to the number of partitions of m into an even number of distinct positive Fibonacci numbers, and $r_O(m)$ is equal to the number of partitions of m into an odd number of distinct positive Fibonacci numbers. We call these partitions "even" and "odd" respectively.

Proposition 1. Let $n \geq 5$ be an integer. Consider the coefficients $a(m)$ for m in the interval $[F_n, F_{n+1})$. Split this interval into the three subintervals $[F_n, F_n + F_{n-3} - 2]$, $[F_n + F_{n-3} - 1, F_n + F_{n-2} - 1]$ and $[F_n + F_{n-2}, F_{n+1} - 1]$.

1. The numbers $a(F_n), a(F_n + 1), \ldots, a(F_n + F_{n-3} - 2)$ are equal to the numbers $(-1)^{n-1}a(F_{n-3} - 2), (-1)^{n-1}a(F_{n-3} - 3), \ldots, (-1)^{n-1}a(0)$ in that order.

2. The numbers $a(F_n + F_{n-3} - 1), a(F_n + F_{n-3}), \ldots, a(F_n + F_{n-2} - 1)$ are equal to 0.

3. The numbers $a(F_n + F_{n-2}), a(F_n + F_{n-2} + 1), \ldots, a(F_{n+1} - 1)$ are equal to the numbers $a(0), a(1), \ldots, a(F_{n-3} - 1)$ in that order.

This description gives a very fast method for computing the coefficients $a(m)$ recursively. Once we have computed them for $0 \le m < F_n$ we can immediately compute them for $F_n \le m < F_{n+1}$ using Proposition 1.

Also, since the coefficient of x^m in $A(x)$ is equal to $-1, 0$ or 1 for all non-negative integers $m < F_5$, it follows inductively that the coefficients in each interval $[F_n, F_{n+1})$ are also all equal to $-1, 0$ or 1. This will prove Robbins's result.

Proof of Proposition 1. It will be convenient to prove Proposition 1.2 first. Let $F_n + F_{n-3} - 1 \le m \le F_n + F_{n-2} - 1$, and consider the partitions of m into distinct positive Fibonacci numbers. It is clear that the largest part in such a partition cannot be F_{n+1} or larger. It cannot be F_{n-2} or smaller either, because $F_{n-2} + F_{n-3} + \cdots + F_2 = F_n - 2 < m$. Therefore, it must be F_n or F_{n-1}.

If the largest part is F_n, then the second largest part cannot be F_{n-1} or F_{n-2}. If, on the other hand, it is F_{n-1}, then the second largest part must be F_{n-2}, because $F_{n-1}+F_{n-3}+F_{n-4}+\cdots+F_2 = 2F_{n-1}-2 = F_n+F_{n-3}-2 < m$.

This means that we can split the set of partitions into pairs. Each pair consists of two partitions of the form $F_n + F_a + F_b + \cdots$ and $F_{n-1} + F_{n-2} + F_a + F_b + \cdots$, where $n - 3 \ge a > b > \ldots$. In each pair, one of the partitions is even and the other is odd. Therefore $r_E(m) = r_O(m)$ and $a(m) = 0$ as claimed.

Now we use a similar analysis to prove Proposition 1.3. Let $F_n + F_{n-2} \le m \le F_{n+1} - 1$. As before, the largest part of a partition of m must be F_n or F_{n-1}. If it is F_n, the second largest part cannot be F_{n-1}. If, on the other hand, it is F_{n-1}, then the second largest part must be F_{n-2}.

Again, we can split *a subset of the set of partitions* into pairs. Each pair consists of two partitions of the form $F_n + F_a + F_b + \cdots$ and $F_{n-1} + F_{n-2} + F_a + F_b + \cdots$, where $n - 3 \ge a > b > \ldots$. In each pair there is an even and an odd partition.

The remaining partitions are of the form $F_n + F_{n-2} + F_a + F_b + \cdots$, where $n-3 \ge a > b > \ldots$. To each one of these partitions we can assign a partition of $m' = m - F_n - F_{n-2}$, by just removing the parts F_n and F_{n-2}. This is in fact a bijection. Since $m' < F_{n-2}$, any partition of m' has largest part less than or equal to F_{n-3}; therefore it can be obtained in that way from a partition of m.

It is clear that, under this bijection, odd partitions of m go to odd partitions of m' and even partitions of m go to even partitions of m'. It follows

that $a(m) = a(m - F_n - F_{n-2})$, as claimed.

Finally we prove Proposition 1.1. Consider $F_n \leq m \leq F_n + F_{n-3} - 2$. The parts of a partition of m come from the list $F_2, F_3, \ldots, F_n$. To each partition π of m, assign the partition π' of $m' = F_{n+2} - 2 - m$ consisting of all the numbers on the above list that do not appear in π. Any partition of m' can be obtained in such a way from a partition of m: the partitions of m' also have all their parts less than or equal to F_n, because it is easily seen that $m' < F_{n+1}$.

So the partitions of m are in bijection with the partitions of m'. If a partition π of m has k parts, the corresponding partition π' of m' has $n - 1 - k$ parts. Therefore, if n is odd, the bijection takes odd partitions to odd partitions and even partitions to even partitions, and $a(m) = a(m')$. If n is even, the bijection takes odd partitions to even partitions, and even partitions to odd partitions, and $a(m) = -a(m')$. In any case, $a(m) = (-1)^{n-1}a(m')$.

Now, it is easily seen that $F_n + F_{n-2} \leq m' \leq F_{n+1} - 2$. Therefore Proposition 1.3 applies, and $a(m') = a(m' - F_n - F_{n-2}) = a(F_n + F_{n-3} - 2 - m)$. Hence $a(m) = (-1)^{n-1}a(F_n + F_{n-3} - 2 - m)$, which is what we wanted to show.

Proposition 2. Given an integer n, pick an integer m uniformly at random from the interval $[0, n]$. Let p_n be the probability that $a(m) = 0$ or, equivalently, that $r_E(m) = r_O(m)$.

Then $\lim_{n\to\infty} p_n = 1$.

Proof. Let α_n be the number of non-zero coefficients among the first F_n coefficents $a(0), a(1), \ldots, a(F_n - 1)$, so that $p_{(F_n-1)} = 1 - \alpha_n/F_n$. Notice that for $F_{n-1} \leq m < F_n$ there are at most α_n non-zero coefficients among $a(0), a(1), \ldots, a(m)$, so $p_m \geq 1 - \alpha_n/(m+1) > 1 - 2\alpha_n/F_n$. We shall now prove that $\lim_{n\to\infty} \alpha_n/F_n = 0$, from which Proposition 2 follows.

First we obtain a recurrence relation for α_n. Consider the non-zero coefficients $a(m)$ for $F_n \leq m \leq F_{n+1} - 1$. We know that there are $\alpha_{n+1} - \alpha_n$ such coefficients. Now split the interval $[F_n, F_{n+1} - 1]$ into the three subintervals $[F_n, F_n + F_{n-3} - 2]$, $[F_n + F_{n-3} - 1, F_n + F_{n-2} - 1]$ and $[F_n + F_{n-2}, F_{n+1} - 1]$. Proposition 1.2 shows that that there are no non-zero coefficients in the second subinterval, and Proposition 1.3 shows that there are α_{n-3} non-zero coefficients in the third subinterval. Because $a(F_{n-3} - 1)$ is non-zero for all $n \geq 5$ (this follows inductively from Proposition 1.3), Proposition 1.1 shows that there are $\alpha_{n-3} - 1$ non-zero coefficients in the first subinterval. We conclude that $\alpha_{n+1} - \alpha_n = 2\alpha_{n-3} - 1$.

The characteristic polynomial of this recurrence relation is $x^4 - x^3 - 2 = 0$, and its roots are approximately $r_1 \approx 1.54, r_2 = -1, r_3 \approx 0.23 + 1.12i$ and $r_4 \approx 0.23 - 1.12i$. It follows from standard results on linear recurrences that $\alpha_n = O(r_1^n)$, while $F_n = \Theta(\lambda^n)$, where $\lambda = (\sqrt{5}+1)/2 \approx 1.62$. Since $r_1 < \lambda$, we conclude that $\lim_{n\to\infty} \alpha_n / F_n = 0$.

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